A renormalization group study of a class of reaction-diffusion models, with particles input

1997 J. Phys. A: Math. Gen. 30 1101

(http://iopscience.iop.org/0305-4470/30/4/013)

J. Phys. A: Math. Gen. **30** (1997) 1101–1114. Printed in the UK PII: S0305-4470(97)77500-0

A renormalization group study of a class of reaction-diffusion models, with particles input

Pierre-Antoine Rey and Michel Droz

Département de Physique Théorique, Université de Genève, CH-1211 Geneva 4, Switzerland

Received 27 August 1996

Abstract. We study a class of reaction-diffusion models extrapolating continuously between the pure coagulation-diffusion case ($A + A \rightarrow A$) and the pure annihilation-diffusion one ($A + A \rightarrow \emptyset$) with particles input ($\emptyset \rightarrow A$) at a rate J. For dimension $d \leqslant 2$, the dynamics strongly depends on the fluctuations while, for $d > 2$, the behaviour is mean-field like. The models are mapped onto a field theory whose properties are studied in a renormalization group approach. Simple relations are found between the time-dependent correlation functions of the different models of the class. For the pure coagulation-diffusion model the time-dependent density is found to be of the form $c(t, J, D) = (J/D)^{1/\delta}\mathcal{F}[(J/D)^{\Delta} Dt]$, where D is the diffusion constant. The critical exponent δ and Δ are computed to all orders in $\epsilon = 2 - d$, where d is the dimension of the system, while the scaling function $\mathcal{F}$ is computed to second order in ϵ. For the one-dimensional case an exact analytical solution is provided whose predictions are compared with the results of the renormalization group approach for $\epsilon = 1$.

1. Introduction

It is by now well established that reaction-diffusion models can have a rich dynamics governed in low dimensions by fluctuations. Several physical quantities behave as power laws and the associated critical exponents have some universal properties. The renormalization group (RG) method developed in the framework of equilibrium statistical physics [1], provides a suitable tool to study such dynamics. In order to be able to approach the problem of reaction-diffusion processes in terms of the RG method, one has first to go from the initial microscopic master equation to a coarsed-grained description. The standard way to do this consists of using a Fock space formalism (see the works of Doi [2], Grassberger and Scheunert [3] and Peliti [4]). One ends up with a model whose dynamics is defined by the action of a continuous field theory. Among several applications, this method has been used to study the two-species annihilation reaction problem $A + B \rightarrow \emptyset$; for a homogeneous initial state with equal densities [5], or unequal densities [6], the rigorous predictions of Bramson and Lebowitz [7] have been reproduced and novel results obtained. For the case in which the two species are initially spatially separated, Howard and Cardy [8] have confirmed scaling arguments developed by Cornell and Droz [9].

Another interesting family of reaction-diffusion processes is formed by the one-species diffusion-annihilation and the diffusion-coagulation models. In 1986, Peliti [10] showed that the coagulation-diffusion model $A + A \rightarrow A$ belongs to the same universality class as the annihilation-diffusion one, $A + A \rightarrow \emptyset$. He also showed that the associated field theory is super-renormalizable. Thus, only the coupling constant needs to be renormalized. Moreover, this renormalization can be performed to all orders in $\epsilon = d_u - d$, where d_u, the

0305-4470/97/041101+14$19.50

upper critical dimension, is 2 in this case. Peliti showed that the concentration $c(t)$ of the reactant in the long time regime behaves as

$$c(t) \propto t^{-\alpha} \qquad \alpha = \frac{d}{2} \qquad (t \to \infty).$$

However, he made no predictions concerning the amplitude, neglecting the initial conditions in its approach. It turns out that the initial conditions may play a very important role. This aspect has been taken into account within this formalism only by Ohtsuki [11] in 1991 and by Friedman *et al* [12] in 1992. In 1994, Lee [13] gave the first complete RG analysis of the annihilation-diffusion model. It turns out that the initial conditions show up as a local source into the action. Lee was able to treat this term to all orders in perturbation theory. It was shown later (see [8] for the two-species annihilation and [14] for the one-species case) that such infinite resummation is equivalent to a shift of the fields in the action by their classical values.

Meanwhile, in 1993, Droz and Sasvári [15] addressed the problem of both annihilation-diffusion and coagulation-diffusion processes in the presence of a source J of particles:

$$A + A \xrightarrow{k} \emptyset \qquad A + A \xrightarrow{g} A \qquad \emptyset \xrightarrow{J} A.$$

Performing a renormalization procedure, they found that the density of particles obeys the scaling law

$$c(t, J, D) = \left(\frac{J}{D}\right)^{d/(d+2)} \mathcal{F}[(J/D)^{2/(d+2)} Dt] \tag{1}$$

for sufficiently small values of J. However, the scaling function $\mathcal{F}$ was not computed. Making *ad hoc* assumptions on the asymptotic behaviour of $\mathcal{F}$ in the limits $t \to \infty$, they showed that the stationary particle density

$$c \sim J^{1/\delta} \qquad \delta = 1 + \frac{2}{d}$$

was approached with a characteristic relaxation time τ given by

$$\tau \sim J^{-\Delta} \qquad \Delta = \frac{2}{d+2}.$$

Moreover, considerations on the $J \to 0$ limit allow them to reproduce the scaling laws postulated phenomenologically by Rácz [16], namely

$$\alpha\delta\Delta = 1 \qquad \Delta + \frac{1}{\delta} = 1.$$

These different exponents are known to all orders in an $\epsilon = 2 - d$ expansion. This is due to the fact that the propagator needs no renormalization and that the coupling constant can be renormalized exactly. Thus, the values of the exponents can be obtained by pure dimensional analysis, independently of the properties of the initial state. In contrast, to compute the scaling function $\mathcal{F}$ it is crucial to take into account the initial condition.

The goal of this paper is two-fold. First, to provide a complete renormalization group analysis of such models by computing not only the exponents but also the scaling function $\mathcal{F}$ defined by (1) in the framework of an ϵ expansion. This is done to first order in ϵ in section 2. Second, to give an exact analytical solution of the one-dimensional coagulation-diffusion model with infinite reaction rate and source by extending to a time-dependent regime an approach developed by Doering and ben-Avraham [17]. These exact results are compared with the RG predictions in the limit $\epsilon = 1$ in section 3. Remarks and conclusions are given in section 4.

2. Field theoretical approach and renormalization group analysis

2.1. The model and the associated field theory

We shall not derive here in detail how one obtains the field theoretical model. The interested reader is referred to the original papers of Doi [2], Grassberger and Scheunert [3] and Peliti [4, 10], or to the short reviews presented in [13] and [18]. As the number of particles is not conserved, the basic idea is to introduce a Fock space representation. The time evolution operator of the problem can then be cast in a path integral form which, in the continuous (coarsed-grained) limit, is characterized by the action:

$$S_\gamma[a,\bar{a},J]=\int \mathrm{d}^d x \int_0^t \mathrm{d}\bar{t}\left[\bar{a}\left(\frac{\partial}{\partial \bar{t}}-D\nabla^2\right)a+\gamma\lambda\bar{a}a^2+\lambda\bar{a}^2a^2-J\bar{a}\right]. \tag{2}$$

A whole class of models indexed by the parameter $\gamma \in [1,2]$ is thus defined. For $\gamma=1$ one has the pure coagulation-diffusion model and for $\gamma=2$ the pure annihilation one. For $1<\gamma<2$, both reactions are possible with a given probability depending on γ (see [18] for more details). The coupling constant λ is related to the reaction rates g and k via $\lambda=(\tilde{\gamma}+1)g\ell^d$ and $\gamma=(\tilde{\gamma}+2)(\tilde{\gamma}+1)^{-1}$, where $\tilde{\gamma}=k/g$ and ℓ is a characteristic microscopic length of the original model. The particles diffuse in an infinite d-dimensional space with a diffusion constant D. The above action could model two different types of colliding particles in some appropriate limits: first, point-like particles living on a d-dimensional hypercubic lattice [13] where, in this case, ℓ is the lattice constant; second, extended particles living in a d-dimensional continuous space [18] and ℓ being the typical size of the particles.

The time- and position-dependent fields a and $\bar{a}$, respectively, obey bosonic-like commutation relations. The field $a(x,t)$ is related to the local particle density, while the auxiliary field $\bar{a}(x,t)$ has no particular physical meaning.

Within this formalism, correlation functions are expressed by functional integrals

$$\begin{aligned}G_\gamma^{N,\bar{N}}(\{x_i,t_i\}_1^{N+\bar{N}})=\mathcal{N}^{-1}\int \mathcal{D}a\,\mathcal{D}\bar{a}\,a(x_1,t_1)\ldots a(x_N,t_N)\\ \times\bar{a}(x_{N+1},t_{N+1})\ldots\bar{a}(x_{N+\bar{N}},t_{N+\bar{N}})\exp(-S_\gamma[a,\bar{a},J])\end{aligned} \tag{3}$$

where $\mathcal{N}=\int \mathcal{D}a\mathcal{D}\bar{a}\exp(-S_\gamma[a,\bar{a},J])$ is a normalization constant, which turns out to be one. Anticipating this fact, we shall now omit it. The particle density $c_\gamma(x,t)$ at point x and time t reads

$$c_\gamma(x,t)=\int \mathcal{D}a\,\mathcal{D}\bar{a}\,a(x,t)\exp(-S_\gamma[a,\bar{a},J]). \tag{4}$$

It can be shown [13, 18] that the correlation functions of different γ models are closely related. In particular, one has

$$S_\gamma[a,\bar{a},J]=S_1[\gamma a,\gamma^{-1}\bar{a},\gamma J]$$

which implies for the concentration

$$c_\gamma(x,t;J)=\gamma^{-1}c_1(x,t;\gamma J). \tag{5}$$

Accordingly, it suffices to study one particular model belonging to the class to know the behaviour of the other members. From now on, we shall study the pure coagulation-diffusion model ($\gamma=1$), with an initial state empty, whose action is

$$S[a,\bar{a},J]=\int \mathrm{d}^d x \int_0^t \mathrm{d}\bar{t}\left[\bar{a}\left(\frac{\partial}{\partial \bar{t}}-D\nabla^2\right)a+\lambda\bar{a}a^2+\lambda\bar{a}^2a^2-J\bar{a}\right]. \tag{6}$$

For the sake of simplicity, we do not write the index $\gamma = 1$ in the following. When $J = \lambda = 0$, one has a free theory (pure diffusion) and the spatial Fourier transform of the free propagator is simply $G_0(p, t) = \theta(t) \exp(-Dp^2 t)$, where $\theta(t)$ is the usual Heaviside function. Simple power counting shows that the upper critical dimension of action (6) is $d_u = 2$. For $d > d_u$, the quadrivertex $\lambda \bar{a}^2 a^2$ is irrelevant. Below d_u, the quadrivertex $\lambda \bar{a}^2 a^2$ is relevant and leads to singularities that have to be renormalized. At $d = d_u$, this vertex is marginal, and one expects logarithmic corrections to the mean-field behaviour.

2.2. Mean-field solution

We first consider the case $d > d_u = 2$. The full action may be replaced by an effective one, without the quadrivertex $\bar{a}^2 a^2$ and in which the coupling λ is replaced by an effective coupling λ_{eff}, depending on the microscopic length ℓ. The behaviour is then mean-field like. The equations of motion for a and $\bar{a}$ are obtained from the effective action, by the usual saddle-point argument and read

$$\frac{\delta S}{\delta \bar{a}} = \left(\frac{\partial}{\partial t} - D\nabla^2 \right) a + \lambda_{\text{eff}}\, a^2 - J = 0 \tag{7}$$

and

$$\frac{\delta S}{\delta a} = -\left(\frac{\partial}{\partial t} + D\nabla^2 \right) \bar{a} + 2\lambda_{\text{eff}}\, \bar{a} a = 0. \tag{8}$$

Assuming that a and $\bar{a}$ are homogeneous, it follows that, as expected, $\bar{a} = 0$ is a solution and (7) becomes

$$\frac{\partial a}{\partial t} = -\lambda_{\text{eff}}\, a^2 + J$$

with the initial condition $a|_{t=0} = 0$. Thus, the mean-field or classical solution is

$$a_{\text{cl}}(t) = \sqrt{\frac{J}{\lambda_{\text{eff}}}} \left(1 - 2 \frac{\exp(-2\sqrt{J\lambda_{\text{eff}}}\, t)}{1 + \exp(-2\sqrt{J\lambda_{\text{eff}}}\, t)} \right) \tag{9}$$

where the subscript 'cl' stands for classical.

2.3. Renormalization

Let us now consider the problem below two dimensions. A brute force computation of the particle density from (4) leads to divergences. Thus, the coupling constant λ needs to be renormalized. Neither the fields a and $\bar{a}$ nor the diffusion constant D require a renormalization [10, 13] and, as a consequence, the particle input rate J does not either.

We define the temporally extended vertex function $\lambda(p, t)$ to be the sum of the diagrams shown in figure 1. These diagrams can be summed to all orders and the Laplace-transformed vertex function reads (see [13])

$$\tilde{\lambda}(p, s) = \frac{\lambda}{1 + 2[\lambda/(8\pi D)^{d/2}]\Gamma(\frac{1}{2}\epsilon)(s + \frac{1}{2}Dp^2)^{-\epsilon/2}}. \tag{10}$$

We define g_R, the dimensionless renormalized coupling (or running coupling constant), using the minimal subtraction scheme. That is we define

$$g_R = Z_g \kappa^{-\epsilon} \frac{\lambda}{2\pi D} \tag{11}$$

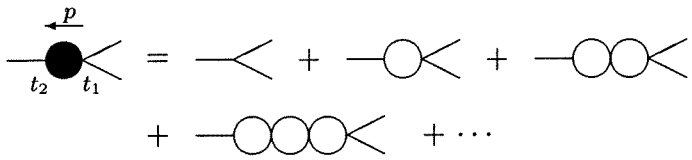

Figure 1. Diagrammatic representation of the temporally extended vertex function $\lambda(p, t_2 - t_1)$. The propagator is represented by a full line. Here the diagrammatic expansion for the trivertex $\lambda\bar{a}a^2$ is shown. Similar diagrams can be drawn for the quadvertex $\lambda\bar{a}^2a^2$.

where κ is a normalization point and $Z_g = 1 + \sum_{i=1}^{\infty} a_i g_R^i$. The a_i are chosen so as to exactly cancel the poles of order $1/\epsilon$ appearing in $\tilde{\lambda}(p, s)$. From (10), we have

$$\tilde{\lambda}(p, s) = \frac{\lambda}{1 + \kappa^{-\epsilon}\lambda[1 + \epsilon A_\epsilon(p, s)]/(2\pi D\epsilon)}$$

where $A_\epsilon(p, s) = O(\epsilon^0)$, for any p and s. With (11), one finds

$$\frac{\tilde{\lambda}(p, s)}{2\pi D} = \frac{\kappa^\epsilon g_R}{Z_g + g_R/\epsilon + g_R A_\epsilon(p, s)}.$$

According to our prescription, we have to choose $a_1 = -1/\epsilon$ and $a_i = 0$ for $i > 1$, giving $Z_g = 1 - g_R/\epsilon$ (exact to all orders). Inserting this result into (11), we find

$$g_R = \frac{\kappa^{-\epsilon}\lambda}{2\pi D}\left(1 + \frac{\kappa^{-\epsilon}\lambda}{2\pi D\epsilon}\right)^{-1}. \tag{12}$$

The computation of the β function is thus straightforward:

$$\beta(g_R) \equiv \kappa\frac{\partial g_R}{\partial \kappa} = -\epsilon g_R + g_R^2.$$

It is exactly quadratic in g_R and has a fixed point given by $\beta(g_R^*) = 0$ at $g_R^* = \epsilon$.

Using (12), we may express the bare coupling in terms of the renormalized one:

$$\kappa^{-\epsilon}\frac{\lambda}{2\pi D} = \frac{g_R}{1 - g_R/g_R^*} = g_R + \frac{g_R^2}{g_R^*} + \cdots. \tag{13}$$

The perturbation theory can then be written as an expansion in powers of g_R.

2.4. Renormalization group equations

An arbitrary renormalized correlation function $G_R^{N,\bar{N}}(\{x_i, t_i\}_1^{N+\bar{N}})$ (where the subscript R stands for renormalized) is related to its bare expression (3) through

$$G_R^{N,\bar{N}}(\{x_i, t_i\}_1^{N+\bar{N}}; g_R, D, J, \kappa) = G^{N,\bar{N}}(\{x_i, t_i\}_1^{N+\bar{N}}; \lambda, D, J).$$

The independence of the bare functions on the normalization scale can be expressed via the condition

$$\left(\kappa\frac{\partial}{\partial\kappa} + \beta(g_R)\frac{\partial}{\partial g_R}\right) G_R^{N,\bar{N}}(\{x_i, t_i\}_1^{N+\bar{N}}; g_R, D, J, \kappa) = 0.$$

The formal solution (obtained by the method of characteristic) is

$$G_R^{N,\bar{N}}(\{x_i, t_i\}_1^{N+\bar{N}}; g_R, D, J, \kappa) = G_R^{N,\bar{N}}(\{x_i, t_i\}_1^{N+\bar{N}}; g_R(\rho), D, J, \rho\kappa) \tag{14}$$

with

$$g_{\mathrm{R}}(\rho) = g_{\mathrm{R}}^{*}\left(1 + \frac{g_{\mathrm{R}}^{*} - g_{\mathrm{R}}}{g_{\mathrm{R}}}\rho^{\epsilon}\right)^{-1}. \tag{15}$$

Note that in the small ρ limit, $g_{\mathrm{R}}(\rho) \to g_{\mathrm{R}}^{*}$.

We can implement (14) with a dimensional analysis. The dimensions of the different quantities, expressed in term of momentum κ and energy E are

$$[t] = E^{-1} \qquad [D] = E\kappa^{-2} \qquad [J] = E\kappa^{d}$$
$$[a] = \kappa^{d} \qquad [\bar{a}] = 1 \qquad [G_{\mathrm{R}}^{N,\bar{N}}(\{x_i, t_i\}_1^{N+\bar{N}})] = \kappa^{Nd}.$$

Thus,

$$G_{\mathrm{R}}^{N,\bar{N}}(\{x_i, t_i\}_1^{N+\bar{N}}; g_{\mathrm{R}}, D, J, \kappa) = \kappa^{Nd} G_{\mathrm{R}}^{N,\bar{N}}(\{\kappa x_i, \kappa^2 D t_i\}_1^{N+\bar{N}}; g_{\mathrm{R}}, 1, \kappa^{-d-2} J/D, 1). \tag{16}$$

The combination of (14) and (16) leads to

$$G_{\mathrm{R}}^{N,\bar{N}}(\{x_i, t_i\}_1^{N+\bar{N}}; g_{\mathrm{R}}, D, J, \kappa)$$
$$= (\rho\kappa)^{Nd} G_{\mathrm{R}}^{N,\bar{N}}(\{\rho\kappa x_i, (\rho\kappa)^2 D t_i\}_1^{N+\bar{N}}; g_{\mathrm{R}}(\rho), 1, (\rho\kappa)^{-d-2} J/D, 1). \tag{17}$$

We can then use the following strategy to compute the correlation functions: first an expansion in power of λ is established; then it is converted into an expansion in power of g_{R} through (13). The singularities in ϵ are eliminated using the renormalization scheme. Now, for a correctly renormalized theory, we can rewrite the g_{R} expansion into an ϵ expansion using (17) and (15). Indeed, introducing the ρ dependence through (17) and letting $\rho \to 0$, $g_{\mathrm{R}}(\rho) \to g_{\mathrm{R}}^{*}$, one obtains $G_{\mathrm{R}}^{N,\bar{N}}$ as an expansion in power of ϵ.

Up to now ρ is an arbitrary parameter, and several choices are possible. For example, if we choose ρ such that

$$(\rho\kappa)^{-d-2}\frac{J}{D} = 1 \tag{18}$$

the limit $\rho \to 0$ becomes equivalent to $J \to 0$: when the source rate is vanishing small, the running coupling approaches its fixed point value.

Another choice is

$$(\rho\kappa)^2 D t_1 = 1$$

and the limit $\rho \to 0$ may be exchanged with $t_1 \to \infty$. However, from (17), we see that $(\rho\kappa)^{-d-2} J/D$ diverges. Thus, with this choice, one should know the behaviour of $G_{\mathrm{R}}^{N,\bar{N}}$ for arbitrary large values of J. However, for large values of J, the ϵ-expansion breaks down (see appendix A). Accordingly, we shall choose ρ according to condition (18) in what follows.

2.5. Density calculation

The density is first calculated using (4), in the framework of a perturbation expansion in power of λ. At the tree level, we find out the mean-field result and we can directly use the RG equation (no renormalization is needed). The first correction to the classical behaviour is given by the one-loop contribution. The corresponding diagram may be calculated using the action obtained from (6) by shifting the field a by its classical value:

$$S[\eta, \bar{\eta}, J] = \int \mathrm{d}^d x \int_0^t \mathrm{d}\bar{t}\left[\bar{\eta}\left(\frac{\partial}{\partial \bar{t}} - D\nabla^2 + 2\lambda a_{\mathrm{cl}}\right)\eta + \lambda\bar{\eta}\eta^2 + \lambda\bar{\eta}^2(a_{\mathrm{cl}}^2 + 2a_{\mathrm{cl}}\eta + \eta^2)\right] \tag{19}$$

where $\eta = a - a_{\rm cl}$ and $\bar{\eta} = \bar{a}$, with $a_{\rm cl}$ given by (9). The free propagator (which is also the classical response function) is

$$G_{\rm cl}(p,t,t') = \theta(t-t')\exp[-Dp^2(t-t')]\left(\frac{\cosh(\sqrt{J\lambda}\,t')}{\cosh(\sqrt{J\lambda}\,t)}\right)^2.$$

Note that because of the initial condition, $G_{\rm cl}$ is not invariant under time translation. Obtaining divergent expressions for the one-loop corrections, we shall renormalize them using the renormalization scheme developed above. We give below a summary of these results.

2.5.1. Tree level. Applying the RG formalism as developed above on the mean-field equation, we find, for J sufficiently small,

$$c_{\rm R}(t; g_{\rm R}, D, J, \kappa) = \frac{1}{\sqrt{2\pi\epsilon}}\left(\frac{J}{D}\right)^{d/(d+2)} \times\left(1 - \frac{2\exp[-2\sqrt{2\pi\epsilon}(J/D)^{2/(d+2)}Dt]}{1+2\exp[-2\sqrt{2\pi\epsilon}(J/D)^{2/(d+2)}Dt]}\right)[1+{\rm O}(\epsilon)] \tag{20}$$

which is universal (independent of $g_{\rm R}$). This result is valid for any time t, because we only need to tune J to be in the critical domain.

For small t we find the expected result

$$c_{\rm R}(t) = Jt + {\rm O}(t^2)$$

and for long time ($t \to \infty$)

$$c_{\rm R}(t) = \frac{1}{\sqrt{2\pi\epsilon}}\left(\frac{J}{D}\right)^{d/(d+2)} \{1 - 2\,\exp[-2\sqrt{2\pi\epsilon}(J/D)^{2/(d+2)}Dt]\}[1+{\rm O}(\epsilon)]. \tag{21}$$

The steady-state value is thus given by

$$c_{\rm R}(\infty) = \frac{1}{\sqrt{2\pi\epsilon}}\left(\frac{J}{D}\right)^{d/(d+2)}[1+{\rm O}(\epsilon)] \tag{22}$$

and it is approached exponentially in time as

$$\delta c_{\rm R}(t) \equiv c_{\rm R}(t) - c_{\rm R}(\infty) \propto \exp[-\gamma_\epsilon (J/D)^{2/(d+2)}Dt] \qquad (t\to\infty) \tag{23}$$

with $\gamma_\epsilon = 2\sqrt{2\pi\epsilon}[1+{\rm O}(\epsilon)]$.

2.5.2. One-loop corrections. The diagram corresponding to the one-loop correction is given in figure 2; its analytic expression is

$$c^{(1)}(t) = \frac{2J\lambda}{\cosh^2(\sqrt{J\lambda}t)}\int_0^t {\rm d}t_2 \int_0^{t_2} {\rm d}t_1 \int \frac{{\rm d}^d p}{(2\pi)^d}\exp[-2Dp^2(t_2-t_1)] \times \frac{\cosh^2(\sqrt{J\lambda}t_1)\sinh^2(\sqrt{J\lambda}t_1)}{\cosh^2(\sqrt{J\lambda}t_2)}.$$

The integral over the momentum gives the factor $[8\pi D(t_2-t_1)]^{-d/2}$. We thus obtain

$$c^{(1)}(t) = \frac{2J\lambda}{(8\pi D)^{d/2}}\frac{t^{1+\epsilon/2}}{\cosh^2(\sqrt{J\lambda}t)}\int_0^1 {\rm d}x_2 \int_0^1 {\rm d}x_1\, x_1^{-1+\epsilon/2} \times \frac{\cosh^2[\sqrt{J\lambda}x_2(1-x_1)]\sinh^2[\sqrt{J\lambda}x_2(1-x_1)]}{\cosh^2(\sqrt{J\lambda}x_2)}.$$

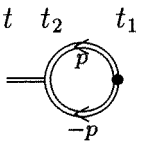

Figure 2. One-loop diagram for the density, using action (19). The double line stands for the free propogator $G_{\rm cl}$ and the dot for the vertex $\lambda a_c^2 I \bar{\eta}^2$.

In the framework of an ϵ-expansion, we eventually find (treating $x_1^{-1+\epsilon/2}$ as a generalized function [20], see appendix B)

$$c^{(1)}(t) = \frac{2J\lambda}{(8\pi D)^{d/2}} \frac{t^{1+\epsilon/2}}{\cosh^2(\sqrt{J\lambda}t)} \left(\frac{2}{\epsilon} \frac{\sinh(2\sqrt{J\lambda}t) - 2\sqrt{J\lambda}t}{2\sqrt{J\lambda}t} + \frac{\phi(\sqrt{J\lambda}t)}{\sqrt{J\lambda}t} + \mathrm{O}(\epsilon) \right)$$

where

$$\phi(\xi) = \xi \int_0^1 \mathrm{d}x_2 \, \ln x_2 \, \sinh^2(\xi x_2) + \xi \int_0^1 \frac{\mathrm{d}x_1}{x_1} \int_0^1 \mathrm{d}x_2 \left(\frac{\cosh^2[\xi x_2(1-x_1)] \sinh^2[\xi x_2(1-x_1)]}{\cosh^2(\xi x_2)} - \sinh^2(\xi x_2) \right).$$

Putting together the mean-field result and the one-loop correction, we verify that the density is indeed divergence free, and using the RG equation (17) we find, for small values of J,

$$\begin{aligned} c_{\rm R}(t; g_{\rm R}, D, J, \kappa) &= \frac{1}{\sqrt{2\pi\epsilon}} \left(\frac{J}{D} \right)^{d/(d+2)} \left\{ \tanh[\sqrt{2\pi\epsilon}\,(J/D)^{2/(d+2)} Dt] \right. \\ &\quad + \frac{\epsilon}{2} \left[\frac{\phi[\sqrt{2\pi\epsilon}(J/D)^{2/(d+2)} Dt]}{\cosh^2[\sqrt{2\pi\epsilon}(J/D)^{2/(d+2)} Dt]} + \frac{1}{2} \ln[8\pi (J/D)^{2/(d+2)} Dt] \right. \\ &\quad \left. \left. \times \left(\tanh[\sqrt{2\pi\epsilon}(J/D)^{2/(d+2)} Dt] - \frac{\sqrt{2\pi\epsilon}(J/D)^{2/(d+2)} Dt}{\cosh^2[\sqrt{2\pi\epsilon}(J/D)^{2/(d+2)} Dt]} \right) \right] + \mathrm{O}(\epsilon^2) \right\} \\ &= \left(\frac{J}{D} \right)^{d/(d+2)} \mathcal{F}[(J/D)^{2/(d+2)} Dt]. \end{aligned} \tag{24}$$

We immediately identify the scaling function $\mathcal{F}$ defined in (1). Taking the limit $t \to \infty$ we find

$$c_{\rm R}(\infty; g_{\rm R}, D, J, \kappa) = \frac{1}{\sqrt{2\pi\epsilon}} \left(\frac{J}{D} \right)^{d/(d+2)} \left\{ 1 - \frac{\epsilon}{4} \left[\gamma_{\rm E} - \ln\left(\frac{2\pi}{\epsilon} \right) \right] + \mathrm{O}(\epsilon^2) \right\} \tag{25}$$

where $\gamma_{\rm E}$ is the Euler constant ($\gamma_{\rm E} \simeq 0.5772$). Unfortunately, due to the complicated form of ϕ, we are unable to give a more compact form for the asymptotics of $\mathcal{F}$. Note that the empty initial condition implies that $\lim_{J \to 0} c_{\rm R}(t) = 0$.

Let us consider two particular cases of interest. The first one is the case $\epsilon = 1$ ($d = 1$). Then the steady-state density is

$$c_{\rm R}(\infty) = \frac{1}{\sqrt{2\pi}} \left(\frac{J}{D} \right)^{1/3} [1 - \tfrac{1}{4}(\gamma_{\rm E} - \ln 2\pi) + \cdots] \tag{26}$$

and it is asymptotically approached (at the tree level) as

$$\delta c_{\rm R}(t) \propto \exp[-\sqrt{8\pi}(J/D)^{2/3} Dt]. \tag{27}$$

We shall compare the accuracy of these expressions with exact results in the next section. Note, however, that, in principle, nothing ensures us that the terms we neglected are small.

The second case is $\epsilon = 0$ ($d = 2$). The running coupling is given by

$$g_{\mathrm{R}}(\rho) = \frac{g_{\mathrm{R}}}{1 - g_{\mathrm{R}} \ln \rho}$$

which goes to $-(\ln \rho)^{-1}$ when $\rho \to 0$. By replacing this expression in our previous formula, we find for the steady-state density

$$c_{\mathrm{R}}(\infty) = \left(\frac{2J}{\pi D}\right)^{1/2} [(\ln J)^{-1/2} + \mathrm{O}((\ln J)^{-1})] \tag{28}$$

and

$$\delta c_{\mathrm{R}}(t) \propto \exp[-\sqrt{8\pi}(J/D)^{1/2} \ln(J) Dt] \tag{29}$$

as anticipated, logarithmic corrections to the mean-field result are obtained.

3. Exact results in one dimension

A large amount of work has been done to solve exactly one-species diffusion reaction models in one dimension (see, for example, [21]). In particular, the diffusion-annihilation and the diffusion-coagulation reactions have been considered with an input of particles. In the diffusion-annihilation case, Rácz [22] obtained the steady-state concentration by mapping its model to the kinetic Glauber-Ising model [23]. In 1988, Doering and ben-Avraham [24] calculated the time-dependent concentration exactly for a simple diffusion-coagulation model, using the interparticle distribution function. Since, their method has been generalized to other diffusion-coagulation processes (see, for example, [25]) and in particular the steady-state concentration has been obtained [17] for the diffusion-coagulation with an input source of particle.

In this section we aim at testing the validity of the RG predictions for $d = 1$. For this purpose we shall extend Doering's and ben-Avraham's results and compute the time-dependent concentration.

We consider an infinite chain (our one-dimensional space) initially empty, and we allow particles to appear randomly at rate J (per unit time and per unit length). Thus initially

$$\left.\frac{\mathrm{d}c(t)}{\mathrm{d}t}\right|_{t=0} = J$$

where $c(t)$ is as before the particle concentration. The particles diffuse on the line (with a diffusion constant D) and when two particles meet they instantaneously coagulate ($A + A \to A$). Note that this model is the same as the pure diffusion-coagulation process of section 2, but with an infinite reaction rate λ. Of course, for such a reaction rate, the perturbation expansion in power of λ is meaningless and one may argue that the two models are not equivalent. However, by examining the relation (12) between the renormalized coupling g_{R} and λ, we see that when λ is infinite, $g_{\mathrm{R}} = g_{\mathrm{R}}^*$. We argue that the infinite reaction rate limit may be obtained by taking the fixed point coupling limit ($g_{\mathrm{R}} \to g_{\mathrm{R}}^*$), that is to say by taking the $\lambda \to \infty$ limit *after* having performed the path integrals. This will be confirmed in the following, at least in one dimension.

In a one-dimensional space, the particle concentration can be related to the probability $E(x, t)$ that an interval of length $x \geqslant 0$ is empty at time t, via

$$c(t) = -\left.\frac{\partial E(x,t)}{\partial x}\right|_{x=0}.$$

As shown by Doering and ben-Avraham [17], $E(x,t)$ has the advantage of obeying a closed equation of evolution, namely

$$\frac{\partial E}{\partial t} = 2D\frac{\partial^2 E}{\partial x^2} - JxE \tag{30}$$

with the two conditions

$$E(0,t) = 1 \qquad E(\infty,t) = 0. \tag{31}$$

From this equation, one can immediately obtain the steady state, by setting the left-hand side to zero. One then recognizes the Airy equation, whose solution is (taking into account conditions (31))

$$E(x,\infty) = \frac{\mathrm{Ai}((J/2D)^{1/3}x)}{\mathrm{Ai}(0)}$$

where $\mathrm{Ai}(z)$ is the Airy function (see [26]). As a consequence, the asymptotic concentration reads

$$c(\infty) = -\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}\left(\frac{J}{2D}\right)^{1/3}. \tag{32}$$

$\mathrm{Ai}'(z)$ is the first derivative of $\mathrm{Ai}(z)$. Note that $\mathrm{Ai}'(0) < 0$.

Before comparing (32) with the RG results, we shall compute the time-dependent part of the concentration. For this purpose, we shall solve (30) using the Laplace transform $\tilde{E}(x,s)$ defined by $\tilde{E}(x,s) = \int_0^t \mathrm{d}t\, \mathrm{e}^{-st}E(x,t)$. Equation (30) becomes

$$2D\frac{\partial^2 \tilde{E}(x,s)}{\partial x^2} - (Jx+s)\tilde{E}(x,s) + E(x,0) = 0 \tag{33}$$

where $E(x,0)$ is the initial condition (for an empty system, $E(x,0) = 1$). Equation (33) is an inhomogeneous second-order ordinary differential equation. Its general solution is the sum of the homogeneous solution $\tilde{E}_\mathrm{h}(x,s)$ and a particular solution. The homogeneous solution is

$$\tilde{E}_\mathrm{h}(x,s) = \left(\frac{J}{2D}\right)^{-2/3}[\alpha_1(s)\mathrm{Ai}(z) + \alpha_2(s)\mathrm{Bi}(z)]$$

where $z = (J/2D)^{-2/3}(Jx+s)/2D$, $\alpha_1(s)$ and $\alpha_2(s)$ are two unknown functions of s and $\mathrm{Bi}(z)$ is the second Airy function [26].

Writing $\tilde{E}(x,s) = (J/2D)^{-2/3}A(z,s)$, (33) becomes

$$\frac{\partial^2 A(z,s)}{\partial z^2} - zA(z,s) = -\frac{1}{2D}$$

for which a well known solution is

$$A(z,s) = -\frac{\pi}{2D}\int_0^z \mathrm{d}v[\mathrm{Ai}(v)\mathrm{Bi}(z) - \mathrm{Ai}(z)\mathrm{Bi}(v)].$$

The two boundary conditions (31) permit us to determine the two unknown functions $\alpha_1(s)$ and $\alpha_2(s)$. We eventually find for the general solution of (33)

$$\tilde{E}(x,s) = \frac{\pi}{2D}\left(\frac{J}{2D}\right)^{-2/3}\left\{\mathrm{Ai}(\xi+\sigma)\int_\sigma^{\xi+\sigma}\mathrm{d}v\,\mathrm{Bi}(v) + \mathrm{Bi}(\xi+\sigma)\left[\frac{1}{3} - \int_0^{\xi+\sigma}\mathrm{d}v\,\mathrm{Ai}(v)\right]\right.$$
$$\left. + \frac{\mathrm{Ai}(\xi+\sigma)}{\mathrm{Ai}(\sigma)}\left[\frac{1}{\pi\sigma} + \mathrm{Bi}(\sigma)\left(\int_0^\sigma \mathrm{d}v\,\mathrm{Ai}(v) - \frac{1}{3}\right)\right]\right\} \tag{34}$$

where $\xi = (J/2D)^{1/3}x$ and $\sigma = (J/2D)^{-2/3}s/2D$.

The probability $E(x,t)$ is then obtained by Laplace inverting (34). For $t > 0$, we only have to care for the poles of $\tilde{E}(x,s)$. They are located at $\sigma = 0$ and at $\sigma = a_n$, $n = 1, 2, 3, \ldots$, where a_n is the nth zero of $\mathrm{Ai}(x)$ ($a_n < 0$). We finally obtain

$$E(x,t) = \frac{\mathrm{Ai}(\xi)}{\mathrm{Ai}(0)} + \sum_{n=1}^{\infty} \frac{\mathrm{Ai}(\xi + a_n)}{\mathrm{Ai}'(a_n)} \left[\frac{1}{a_n} + \pi \mathrm{Bi}(a_n) \left(\int_0^{a_n} \mathrm{d}v\, \mathrm{Ai}(v) - \frac{1}{3} \right) \right] \mathrm{e}^{-|a_n|\tau}$$

with $\tau = 2Dt(J/2D)^{2/3}$ and

$$c(t) = \left(\frac{J}{2D} \right)^{1/3} \left\{ - \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} - \sum_{n=1}^{\infty} \left[\frac{1}{a_n} + \pi \mathrm{Bi}(a_n) \left(\int_0^{a_n} \mathrm{d}v\, \mathrm{Ai}(v) - \frac{1}{3} \right) \right] \mathrm{e}^{-|a_n|\tau} \right\}. \tag{35}$$

We are now in position to compare these results with the RG results for $\epsilon = 1$. For the steady state, the RG method gives (up to the one-loop corrections) (26)

$$c_{\mathrm{R}}(\infty) \simeq 0.53 \left(\frac{J}{D} \right)^{1/3} \tag{36}$$

(for small J) while (putting $t = \infty$ inside (35)) the exact solution gives

$$c(\infty) = - \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} \left(\frac{J}{2D} \right)^{1/3} \simeq 0.58 \left(\frac{J}{D} \right)^{1/3} \tag{37}$$

(for arbitrary J). Surprisingly, the difference is only of the order of 10%.

The comparison for the approach to the steady state is less convincing, mainly due to the fact that we do not know the one-loop corrections. The RG method gives, from equation (27),

$$\begin{aligned} \delta c_{\mathrm{R}}(t) &= - \left[1 - \frac{1}{4} (\gamma_{\mathrm{E}} - \ln 2\pi) \right] \left(\frac{2}{\pi} \right)^{1/2} \left(\frac{J}{D} \right)^{1/3} \exp[-\sqrt{8\pi} (J/D)^{2/3} Dt] \\ &\simeq -1.06 \left(\frac{J}{D} \right)^{1/3} \exp[-5.01 (J/D)^{2/3} Dt] \end{aligned} \tag{38}$$

whereas the exact result is

$$\begin{aligned} \delta c(t) &= \left[\frac{1}{a_1} + \pi \mathrm{Bi}(a_1) \left(\int_0^{a_1} \mathrm{d}v\, \mathrm{Ai}(v) - \frac{1}{3} \right) \right] \left(\frac{J}{2D} \right)^{1/3} \mathrm{e}^{-|a_1|\tau} \\ &\simeq -1.10 \left(\frac{J}{D} \right)^{1/3} \exp[-2.95 (J/D)^{2/3} Dt] \end{aligned} \tag{39}$$

(with $a_1 \simeq -2.33$). Both amplitudes in front of the exponential are in good agreement (because we used the one-loop result of the steady state). However, the tree-level amplitude into the exponential is quite different (of almost 70%) from the one given by the exact theory. The inclusion of the one-loop correction should lead to a better agreement.

Note that the exact results are valid without any restriction on J, in contrast to the RG results, which apply only for small J. This restriction was introduced to ensure that g_{R} is inside the critical domain (i.e. g_{R} near g_{R}^*). In view of the predictions of the exact theory, it turns out that this restriction over J is unnecessary. The coupling stays inside the critical domain for any value of J. This justifies why, when $\lambda \to \infty$, $g_{\mathrm{R}} \to g_{\mathrm{R}}^*$. We can make this point clearer. Indeed, expanding the running coupling constant (15) for small J, we can identify

$$J_{\mathrm{c}} = D \left(\frac{\lambda}{2\pi D \epsilon} \right)^{(d+2)/(2-d)}$$

as a crossover scale for J. The critical domain is thus obtained when $J \ll J_{\mathrm{c}}$. This scale grows when λ grows or when ϵ decreases.

4. Discussion and conclusive remarks

We have shown that the RG method is a suitable formalism to compute the density in a wide class of diffusion-reaction models, with an input of particles. In particular, we have easily calculated the critical exponents in arbitrary dimension. However, the computation of the universal scaling function is generally much more difficult (see, for example, (24)) and can only be achieved within a power expansion in $\epsilon = 2-d$. However, the comparison of the first-order results for $\epsilon = 1$ are not too far from the results obtained exactly in one dimension by a different approach.

One of the main interests of the RG method is that, contrary to most unidimensional exact approaches, the RG approach is not restricted to the computation of the particle density and higher correlation functions can also be calculated (although the computation can become quite involved, see [13]). Moreover, the RG approach provides a framework in which universality can be established. This is not the case of an exact solution obtained for particular models.

Another advantage of the RG approach is the fact that the properties of a whole class of model can be simply related. This allows us using (5), and knowing the particle density for the diffusion-coagulation model, to obtain the density of any mixed annihilation-coagulation process. In particular, for the pure diffusion-annihilation model ($\alpha = 2$) in one dimension, one recovers the steady-state density previously calculated by Rácz [22].

The present work can be generalized in several directions. Generalization to $mA \to \emptyset$ ($m > 2$) reactions with a source of particles is also straightforward. Indeed, following Lee [13], it turns out that apart the lowering of the critical dimension to $d_u = 2/(m-1)$, only minor changes occur (for example, the amplitudes become m-dependent). This is due to the fact that in our renormalization scheme only the fixed point g_R^* of the coupling is modified. The structure of the equations remain unchanged and, in particular, the RG equation (17) still holds. As a consequence, the critical exponents below d_u are the same as for $m = 2$. The steady-state concentration can be written (for $d < d_u$) as

$$c_{m,R}(\infty) = A_m \left(\frac{J}{D}\right)^{d/(d+2)}$$

and the universal amplitude A_m can be computed within an ϵ-expansion. The approach to the steady state is still exponential with k-dependent amplitudes. Extension to the reactions $mA \to lA$ with $l < m$ ($m > 2$) is also possible (see [13]).

Another natural extension of this work is to consider the possibility of reversible reaction (for example, $A + A \rightleftharpoons A$). For such a system a completely different physics is expected. Indeed, a quick investigation of the associated action shows that the upper critical dimension is no longer 2 but 4. In addition, it appears that a wavefunction renormalization is needed, giving rise to anomalous dimensions. The computation of scaling functions for this problem is currently under investigation.

Appendix A

We aim to show that in the limit $J \to \infty$, the ϵ-expansion breaks down. For this purpose, let us consider the action (19) obtained by the shift of the field a by its classical value. The shift permits us to suppress the source term in the action; in other words, shifting the field is equivalent to perform the infinite sum generated by the source term. One could then think that any quantity can be calculated for arbitrary J. This is, however, not true for the following reason. To calculate a given quantity, we expand the action with respect to

the $\lambda\bar{\eta}^2(a_{\rm cl}^2 + 2a_{\rm cl}\eta + \eta^2)$ term. We shall then obtain a power expansion in term of λ with coefficient proportional to $a_{\rm cl}^2$. However, for large J, $a_{\rm cl}$ tends to $(J/\lambda)^{1/2}$. The expansion in λ is partly replaced by an expansion in J, which does not lead to an ϵ-expansion. The only possibility to revert this would be to treat non-perturbatively the action, which is out of the question.

Appendix B

The problem is to compute the ϵ-expansion of

$$\int_0^1 {\rm d}x\, x^{-1+\epsilon} f(x). \tag{B1}$$

Putting naively $x^{-1+\epsilon} = x^{-1}[1 + \epsilon \ln x + {\rm O}(\epsilon^2)]$ in (B1) clearly fails, because if $f(0) \neq 0$, $\int_0^1 {\rm d}x\, x^{-1} f(x)$ diverges. One way to avoid this problem is to treat $x^{-1+\epsilon}$ as a generalized function. We shall not give here a detailed discussion of the generalized functions (see [20] for an introduction). We only quote the result, namely

$$\int_0^1 {\rm d}x\, x^{-1+\epsilon} f(x) = \frac{1}{\epsilon} f(0) + \int_0^1 {\rm d}x\, x^{-1}[f(x) - f(0)] + \epsilon \int_0^1 {\rm d}x\, x^{-1} \ln x [f(x) - f(0)] + {\rm O}(\epsilon^2).$$

Note that there is a pole of order $1/\epsilon$.

References

[1] Wilson K G and Kogut J 1974 *Phys. Rep.* C **12** 75
[2] Doi M 1976 *J. Phys. A: Math. Gen.* **9** 1465
Doi M 1976 *J. Phys. A: Math. Gen.* **9** 1479
[3] Grassberger P and Scheunert M 1980 *Fortschr. Phys.* **28** 547
[4] Peliti L 1985 *J. Physique* **46** 1469
[5] Lee B P and Cardy J 1995 *J. Stat. Phys.* **80** 971
[6] Rey P-A and Droz M 1996 unpublished
[7] Bramson M and Lebowitz J 1991 *J. Stat. Phys.* **62** 297
[8] Howard M and Cardy J 1995 *J. Phys. A: Math. Gen.* **28** 3599
[9] Cornell S and Droz M 1993 *Phys. Rev. Lett.* **70** 3824
[10] Peliti L 1986 *J. Phys. A: Math. Gen.* **19** L365
[11] Ohtsuki T 1991 *Phys. Rev.* A **43** 6917
[12] Friedman B, Levine G and O'Shaughnessy B 1992 *Phys. Rev.* A **46** R7343
[13] Lee B P 1994 *J. Phys. A: Math. Gen.* **27** 2633
[14] Rey P-A 1994 unpublished
[15] Droz M and Sasvári L 1993 *Phys. Rev.* E **48** R2343
[16] Rácz Z 1985 *Phys. Rev.* A **32** 1129
[17] Doering C R and ben-Avraham D 1989 *Phys. Rev. Lett.* **62** 2563
[18] Balboni D, Rey P-A and Droz M 1995 *Phys. Rev.* E **52** 6220
[19] Amit D J 1984 *Field Theory, the Renormalization Group, and Critical Phenomena* (Singapore: World Scientific)
[20] Gel'fand I M and Shilov G E 1964 *Generalized functions* vol 1 (New York: Academic) p 84ff
[21] Alcaraz F C, Droz M, Henkel M and Rittenberg V 1994 *Ann. Phys., NY* **230** 250
Lushnikov A A 1987 *Phys. Lett.* **120A** 135
Balding D, Clifford P and Green N J B 1988 *Phys. Lett.* **126A** 481
Spouge J L 1988 *Phys. Rev. Lett.* **60** 871
Privman V, Cadilhe A M R and Glasser M L 1995 *J. Stat. Phys.* **81** 881
Grynberg M D and Stinchcombe R B 1995 *Phys. Rev. Lett.* **74** 1242

[22] Rácz Z 1985 *Phys. Rev. Lett.* **55** 1707
[23] Glauber R J 1963 *J. Math. Phys.* **4** 294
[24] Doering C R and ben-Avraham D 1988 *Phys. Rev.* A **38** 3035
[25] ben-Avraham D, Burschka M A and Doering C R 1990 *J. Stat. Phys.* **60** 695
Doering C R, Burschka M A and Horsthemke W 1991 *J. Stat. Phys.* **65** 953
[26] Antosiewicz H A 1972 *Handbook of Mathematical Functions* edited by M Abramowitz and I A Stegun (National Bureau of Standards) p 446ff